A FORMULA FOR PRIME NUMBERS

WITOLD A. KOSSOWSKI

ABSTRACT. The paper presents a general formula for prime numbers, as well as its recursive form.
The formula is operational and any number of successive primes can be calculated without any time-consuming operation as checking, division, multiplying, etc.

This time,
when playing at ducks and drakes
ever throw the first stone
found in the calm water.

Let us denote the set of odd integers by O, the set of odd prime numbers by *P* and the set of prime numbers by *S*.

Definition 1. Let $\delta : Z^+ \rightarrow$ O be defined by $\delta a = 2a + 1$ and let O$\rightarrow$ Z$^+$ be defined by $\beta b = ½(b - 1)$.
We will say that *a* **generates** δa.

Theorem 1. Let $q, a, b \in O$ and $i, j \in N$.
For any $k = i+j\ \delta i = j+i\delta j = i+2ij+j$ there is $\delta k=\delta i \delta j,$
as well as for any $q = ab$, there is $\beta q = \beta a + a\beta b = \beta b + b\beta a = \beta a + 2\beta a\beta b + \beta b.$

Proof.
$\delta k=2(i+2ij+j)+1=(2i+1)(2j+1)=\delta i\delta j.$
On the contrary, if $q = ab,$ then $\beta q = ½\ (ab - 1) = ½\ ((2\beta a + 1)(2\beta b + 1) - 1) =$
$= \beta a + 2\beta a\beta b + \beta b = \beta a + a\beta b = \beta b + b\beta a.$
(So, $\beta q \equiv \beta a \pmod a$ and $\beta q \equiv \beta b \pmod b$.)

Example 1.
$\beta 147=73=10 + 2\bullet 10\bullet 3+3=24 + 2\bullet 24\bullet 1+1$
Thus, $147=\delta 10\bullet \delta 3=21\bullet 7$, as well as $147= \delta 24\bullet \delta 1=49\bullet 3$.

Definition 2. A number *k,* which cannot be written in the form $k = i + j\delta i,$ where $i, j \in N,$ will be called a **stone**. The set of stones will be denoted by *K* and the set of stones in range from *m* to *n* by $\overset{n}{\underset{m}{K}}$.

Example 2.

$\overset{28}{\underset{1}{K}}$ = {1, 11,2,3,5,6,8,9,11,14,15,18,20,21,23,26 }.

According to Theorem 1, we can say that:

Remark 1.
(a) The stones generate prime numbers and every odd prime number is generated by a stone, i.e. the set of odd prime numbers is $P = \delta K$.
(b) For any number m and for all successive $n \in N$, we get a sequence
$\delta m_n = \delta (m + n\delta m)$, where every term δm_n can be divided by a number δm, and by an odd number δn.
Consequently, for any stone k and for all successive $n \in N$, we get a sequence
$\delta k_n = \delta (k + n\delta k)$, where every term δk_n can be divided by a prime number δk and by an odd number δn.
(c) **Attention:** Number m is absent in sequence $m + n\delta m$, $n = 1, 2, 3, \ldots$.

Example 3.
for n = 1, 2, 3, ... and m = 4 the sequence $m + n\delta m = 4 + 9n$ = 13, 22, 31, 40, 49, 58, … .

Definition 3. We will say that the numbers $m_n = m + n\delta m$, $n \in N$, are **touched** by number m. The set $\{m_n\}$, composed of all numbers touched by m, *will* be called a set M_m.
We will denote by δM_m the set of numbers generated by the elements of set M_m.

It follows immediately from Theorem 1 that:

Remark 2.
Set δM_m is a set of all odd numbers divisible by δm, **greater** than δm.
Consequently, there are not stones in aby M_m and $\bigcup_{m=1}^{\infty} M_n$ contains generators of all odd numbers with the exception of primes.
So, the set of stones equals $K = N - \bigcup_{m=1}^{\infty} M_n$.

Example 4.
M_1 = {4,7,10,13,16,19,...} and δM_1 = {9,15,21,27,33,39, ...}
M_2 = {7,12,17,22,27,32, ...} and δM_2 = {15,25,35,45,55,65, ...}
M_3 = {10,17,24,31,38,45, ...} and δM_3 = {21,35,49,63,77,91, ...}
As we can see, the set δM_1 contains all odd numbers greater than 3 and divisible by 3,
the set δM_2 contains all odd numbers greater than 5 and divisible by 5, etc..

As we can verify, there are no stones in the set $\bigcup_{m=1}^{\infty} M_n$ = {4, 7, 10, 12, 13, 16, 17, 19, 22, .. }

and no primes in $\delta \bigcup_{m=1}^{\infty} M_n$ = {9, 15, 21, 25, 27, 33, 35, 39, 45, ... }.

Now, we dispose of all the means needed to give the formula for the set S of prime numbers:

$$(1)\qquad S = \delta(N - \bigcup_{m=1}^{\infty} M_m) \cup \{2\}, \quad where\, M_m = \{ m + n\delta m \},\ n = 1, 2, 3, \ldots, \infty$$

Remark 3.
With reference to Definition 2 we can say that if $m = i + 2ij + j$, where $i, j \in N$, then $M_m \subset M_i$ and $M_m \subset M_j$. Thus, the elements M_m in $\bigcup_{m=1}^{\infty} M_n$ are redundant for all m that are not stones and we can impose $m : m \in K$.

Set $K = N - \bigcup_{m=1}^{\infty} M_n$ can be determined step by step, using the recursive definition of the set, with $K_1 = N$ as the set of fundamental objects. We get:

$$(2)\qquad K_m = K_{m-1} - M_{m-1}, \quad m \to \infty, \quad m \in K_{m-1}$$

Example 5.
We start with $K_1 = N = \{1,2,3, \ldots \}$
for $m = 1$, $K_2 := K_1 - M_1 = K_1 - \{ 4, 7, 10, 13, 16, 19, 22, 25, 28, 31, 34, 37, 40, 43, 46, 49, 52, 55, 58, \ldots \} =$
$= \{ 1, 2, 3, 5, 6, 8, 9, 11, 12, 14, 15, 17, 18, 20, 21, 23, 24, 26, 27, 29, 30, 32, 33, 35, 36, 38, 39, 41, 42, 44, 45, 47, 48, 50, 51, 53, 54, 56, 57, 59, 60, \ldots \}$
for $m = 2$, $K_3 := K_2 - M_2 = K_2 - \{ 7, 12, 17, 22, 27, 32, 37, 42, 47, 52, 57, \ldots \} =$
$= \{ 1, 2, 3, 5, 6, 8, 9, 11, 14, 15, 18, 20, 21, 23, 24, 26, 29, 30, 33, 35, 36, 38, 39, 41, 44, 45, 48, 50, 51, 53, 54, 56, 59, 60.... \}$
for m = 3, $K_4 := K_3 - M_3 = K_3 - \{ 10, 17, 24, 31, 38, 45, 52, 59, \ldots \} =$
$= \{ 1,2,3,5,6, 8, 9, 11, 14, 15, 18, 20, 21, 23, 26, 29, 30, 33, 35, 36, 39, 41, 44, 48, 50, 51, 53, 54,56, 60, \ldots \}$
for $m = 4$, $K_5 := K_4$, *as* $m = 4 \notin K_4$
for $m = 5$, $K_6 := K_5 - M_5 = K_5 - \{ 16, 27, 38, 49, 60, \ldots \} =$
$= \{ 1, 2, 3, 5, 6, 8, 9, 11, 14, 15, 18, 20, 21, 23, 26, 29, 30, 33, 35, 36, 39, 41, 44, 48, 50, 51, 53, 54, 56, \ldots \}$
. . .

Theorem 2. When stopping procedure (2) at any m, the range d of determined stones is

$$d = < 1,\ 2m(m + 1) >$$

Proof.
In any M_m, the m^{th} element is a generator of m^2, as, according to Remark 1b,
$b(m + m\delta m) = 2(m + m(2m + 1)) + 1 = (2m + 1)^2 = (\delta m)^2$.
All elements of M_m less than $\beta((\delta m)^2) = 2m(m + 1)$ generate the products of m and of all successive odd numbers less than m.
At the same time, any number n less than $2m(m + 1)$ either is a stone or generates the product ab, where $\beta a < m < \beta b$, what means that n has already been touched by βa.

Example 6.

Let us return to previous example. If we stop the procedure at *m* = 5, a part of already determined set *K* contains stones from the range < 1, 2•5•6> = < 1,60>.

$\overset{60}{\underset{1}{K}}$ = { 1, 2, 3, 5, 6, 8, 9, 11,14,15, 18, 20, 21, 23, 26, 29, 30, 33, 35, 36, 39, 41, 44, 48, 50, 51, 53, 54, 56 }

So, the set of prime numbers in the range < 1, δ60 > = < 1, 121 > is as follows:

$\overset{121}{\underset{1}{S}} = \delta \overset{60}{\underset{1}{K}} \cup \{2\}$ = { 2, 3, 5, 7,11, 7 3, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97, 101, 103, 107,109, 113}

Remark 4.

Obviously, to determine the set $\overset{d}{\underset{1}{K}}$, we can find the m_{max} from the inequality $2m(m + 1) \leq d$ and execute procedure (2) from 1 to m_{max}.

Example 7.

To find all primes from the range < 1, 61 >, we will take *d* = β61 = 30 and *m* = 3, calculated from the inequality $2m(m + 1) \leq 30$.

Final remark

In any calculation, according to procedure (1) or (2), lists or tables (with simple flags) may be useful.

Practically, the only operation to execute is moving the cursor (index) that corresponds to **touching.** There is no need of checking, multiplying nor division to determine the primes in any range.

WITOLD A. KOSSOWSKI
66 RUE ANDRE CHENIER
91300 MASSY
FRANCE

E-mail address : perso@kossowski-witold.eu